STAT547C: Topics in Probability for Statistics	Winter Term 1 (Fall) 2022-23

Assignment 5

Due date: Nov. 25, 11:59 pm. — *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. *Justify your answers formally.*
2. Submit your work by 11:59 pm (Vancouver time) on the due date to Canvas (.pdf).
3. Questions worth more than 1 point will be graded for correctness (partial credit will be given, so make it clear what you're doing and why). The remaining questions will be marked on binary scale for "Did you make a good effort?"

1 Jumping distribution functions

Let F be a distribution function. Show that in general F can have an infinite number of jump discontinuities, but that there can be at most countably many.

[5 mark(s)]

2 Fun times with size-biased densities

Let X be a random variable taking values in $\mathbb{R}_+$, with $\mathbb{E}[X] \in (0, \infty)$. Let its distribution be μ. Assume that μ has (continuous almost everywhere on $\mathbb{R}_+$) density function f_X.

Let X^* be another random variable such that its density function is

$$f_{X^*}(x) = \frac{x}{\int_0^\infty s f_X(s) ds} f_X(x) = \frac{x f_X(x)}{\mathbb{E}[X]} \; .$$

X^* is called a *size-biased* version of X.

Note: Assume that we're working with the Borel σ-algebra on $\mathbb{R}_+$.

(a) Show that the density f_{X^*} determines a unique valid probability distribution on $\mathbb{R}_+$.

[1 mark(s)]

(b) Suppose that X is a random variable distributed as Gamma(a, b), with density function

$$f_X(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} , \quad x > 0, a > 0, b > 0 .$$

What is the distribution of X^*, with density function as in part (a)?

[1 mark(s)]

(c) For a random variable Y taking values in $\mathbb{N}$ with $\mathbb{E}[Y] \in (0, \infty)$, we can do something analogous to part (a): If $P(Y = k) = f_Y(k)$ for $k = 0, 1, 2, \ldots$,

$$f_{Y^*}(k) = \frac{k f_Y(k)}{\mathbb{E}[Y]} , \quad k = 0, 1, 2, \ldots$$

uniquely characterizes a valid probability distribution on $\mathbb{N}$.
If Y is distributed as Poisson(λ) with

$$P(Y = k) = f_Y(k; \lambda) = e^{-\lambda} \frac{\lambda^k}{k!} , \quad \lambda > 0, k = 0, 1, 2, \ldots,$$

what is the distribution of Y^*, defined via $(f_{Y^*}(k; \lambda))_{k \geq 0}$ as in part (c)?

[1 mark(s)]

(d) Let Z be a random variable taking values on $\mathbb{N}$, with negative binomial distribution NB(r, p),

$$P(Z = k) = f_Z(k; r, p) = \frac{\Gamma(k + r)}{\Gamma(k + 1)\Gamma(r)} (1 - p)^r p^k , \quad r > 0, p \in (0, 1), k = 0, 1, 2, \ldots$$

(i) Show that we can obtain Z by first sampling a random Λ distributed as Gamma$(r, \frac{1-p}{p})$, and then using that realization of Λ to sample Z from Poisson(Λ). In other words, show that

$$f_Z(k; r, p) = \mathbb{E}[f_Y(k; \Lambda)] , \quad k = 0, 1, 2, \ldots,$$

where the expectation is with respect to Λ.

[1 mark(s)]

(ii) Let Z^* be a random variable with distribution defined by $f_{Z^*}(k; r, p) = k f_Z(k; r, p)/\mathbb{E}[Z]$. Show that

$$f_{Z^*}(k; r, p) = \mathbb{E}[f_{Y^*}(k; \Lambda^*)] , \quad k = 0, 1, 2, \ldots,$$

where the expectation is with respect to Λ^*, which has density

$$f_{\Lambda^*}\left(\lambda; r, \frac{1 - p}{p}\right) = \frac{\lambda f_\Lambda(\lambda; r, \frac{1-p}{p})}{\mathbb{E}[\Lambda]} ,$$

and $f_\Lambda(\lambda; r, \frac{1-p}{p})$ is the density function of the Gamma$(r, \frac{1-p}{p})$ distribution.

[3 mark(s)]

Question total: [12 mark(s)]

Assignment total: [12 mark(s)]