STAT547C: Topics in Probability for Statistics **Winter Term 1 (Fall) 2022-23**

Assignment 4

Due date: Nov. 2, 11:59 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. *Justify your answers formally.*
2. Submit your work by 11:59 pm (Vancouver time) on the due date to Canvas (.pdf).
3. Questions marked **graded** will be graded for correctness (partial credit will be given, so make it clear what you're doing and why). The remaining questions will be marked on binary scale for "Did you make a good effort?"

1 Measures from indefinite integrals

1. Recall that for a measure space $(E, \mathcal{E}, \mu)$ and a measurable function $p : E \to \mathbb{R}_+$, the indefinite integral of p with respect to μ is

$$\nu(A) = \mu(p\mathbf{1}_A) = \int_A \mu(dx)p(x) \; , \quad A \in \mathcal{E} \; .$$

Assume that $\mu p = \int_E \mu(dx)p(x) < \infty$. Show that ν is a finite measure on $(E, \mathcal{E})$, and that if $\mu p = 1$ then ν is a probability measure.

[5 mark(s)]

2. If $g : E \to \mathbb{R}_+$ is $\mathcal{E}$-measurable, show that

$$\nu g = \int_E \nu(dx)g(x) = \int_E \mu(dx)p(x)g(x) = \mu(pg) \; .$$

Note: This is Proposition I.5.6 in Çinlar. Rather than give his proof, prove this directly, first with simple functions and then for positive measurable functions.

[5 mark(s)]

3. Continuing from above, let λ be the Lebesgue measure on $\mathbb{R}$ and let $p\colon \mathbb{R} \to \mathbb{R}_+$ be a positive measurable function with $\lambda p = 1$. Argue that if ν_p is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\nu_p([-\infty, r]) = \int_{-\infty}^{r} \lambda(dx) p(x) , \quad r \in \mathbb{R} , \tag{1}$$

then ν_p is the unique probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that (1) holds.

[1 mark(s)]

4. Now suppose that

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-a)^2} , \quad x \in (-\infty, \infty) ,$$

the Gaussian density with mean a and variance σ^2. Let X be a random variable whose distribution is ν_p as in (1) and p the Gaussian density. What is $\mathbb{E}[\exp(tX)]$?

[1 mark(s)]

Question total: [12 mark(s)]

Assignment total: [12 mark(s)]