STAT547C: Topics in Probability for Statistics **Winter Term 1 (Fall) 2022-23**

Assignment 3

Due date: Oct. 26, 11:59 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. *Justify your answers formally.*
2. Submit your work by 11:59 pm (Vancouver time) on the due date to Canvas (.pdf).
3. Questions marked **graded** will be graded for correctness (partial credit will be given, so make it clear what you're doing and why). The remaining questions will be marked on binary scale for "Did you make a good effort?"

1 Arithmetic of measures

Let $(E, \mathcal{E})$ be a measurable space.

1. Let μ, ν be two measures on $(E, \mathcal{E})$. Show for all $a, b \geq 0$, the function $\rho(A) = a\mu(A) + b\nu(A)$, $A \in \mathcal{E}$, is a measure.

   [1 mark(s)]

2. Let $\mu_1, \mu_2, \ldots$ be countably many measures on $(E, \mathcal{E})$, and let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers. Show that $\mu(A) = \sum_{j=1}^{\infty} \alpha_j \mu_j(A)$, $A \in \mathcal{E}$, is a measure.

   Hint: To show countable additivity use the following lemma: for any double sequence β_{ij}, $i, j \in \mathbb{N}$, of real numbers,

   $$\sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} \beta_{ij} = \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \ .$$

   Thus, $\lim_{i\in\mathbb{N}} \lim_{j\in\mathbb{N}} \beta_{ij} = \lim_{j\in\mathbb{N}} \lim_{i\in\mathbb{N}} \beta_{ij}$ when $(\beta_{ij})_{i\in\mathbb{N}}$ increases for fixed j, and when $(\beta_{ij})_{j\in\mathbb{N}}$ increases for fixed i.

   [1 mark(s)]

Question total: [2 mark(s)]

2 Functional representation of measurable functions

Recall that for a set E and a measurable space $(F, \mathcal{F})$, the σ-algebra $f^{-1}\mathcal{F}$ on E generated by a function $f : E \to F$ is $f^{-1}\mathcal{F} = \{f^{-1}B : B \in \mathcal{F}\}$.

Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, and $(F, \mathcal{B}(F))$ a standard Borel space. Fix two $\mathcal{E}$-measurable functions $f : E \to F$ and $g : E \to G$. Show that f is $g^{-1}\mathcal{G}$-measurable (i.e., $f^{-1}\mathcal{F} \subset g^{-1}\mathcal{G}$) if and only if there exists some $\mathcal{G}$-measurable mapping $h : G \to F$ such that $f = h \circ g$.

Hint: First, prove for indicator and simple functions, then for positive and arbitrary functions.

[5 mark(s)]

Question total: [5 mark(s)]

3 Random variable from your research interests

Give the full definition of a random variable that is not encountered in introductory textbooks on probability, ideally one you encounter in research.

[1 mark(s)]

Assignment total: [8 mark(s)]