STAT547C: Topics in Probability for Statistics **Winter Term 1 (Fall) 2022-23**

Assignment 2

Due date: Oct. 12, 11:59 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. *Justify your answers formally.*
2. Submit your work by 11:59 pm (Vancouver time) on the due date to Canvas (.pdf).
3. Questions marked **graded** will be graded for correctness (partial credit will be given, so make it clear what you're doing and why). The remaining questions will be marked on binary scale for "Did you make a good effort?"

1 Basics of measurability

In all of the following, let E be a set.

1. Let $\mathcal{E}$ be a σ-algebra on E. If $f_1 \colon E \to \mathbb{R}$ and $f_2 \colon E \to \mathbb{R}$ are $\mathcal{E}$-measurable functions, show that $f'(x) = \max\{f_1(x), f_2(x)\}$ is also $\mathcal{E}$-measurable.

   [1 mark(s)]

2. Let $\mathcal{E}$ be a σ-algebra on E. If $f_1 \colon E \to \mathbb{R}$ and $f_2 \colon E \to \mathbb{R}$ are $\mathcal{E}$-measurable functions, show that $f'(x) = \min\{f_1(x), f_2(x)\}$ is also $\mathcal{E}$-measurable.

   [1 mark(s)]

2 Product spaces and sections

This exercise extends basic properties of measurable functions of one variable to functions of multiple variables. Let $(E, \mathcal{E}), (\mathcal{F}, \mathcal{F}), (\mathcal{G}, \mathcal{G})$ be measurable spaces.

A function that is measurable with respect to $\mathcal{E}$ and $\mathcal{F}$ is said to be $\mathcal{E}/\mathcal{F}$-measurable.

For sets $A \subset E$ and $B \subset F$, we let $A \times B = \{(x,y) : x \in A,\ y \in B\}$. If $A \in \mathcal{E}$ and $B \in \mathcal{F}$, then $A \times B$ is called a **measurable rectangle**. For the **product space** $E \times F$, the **product σ-algebra**, denoted $\mathcal{E} \otimes \mathcal{F}$, is the σ-algebra generated by the collection of measurable rectangles. The resulting measurable space is $(E \times F, \mathcal{E} \otimes \mathcal{F})$.

1. Let $f : E \to F$ be $\mathcal{E}/\mathcal{F}$-measurable, and $g : E \to G$ be $\mathcal{E}/\mathcal{G}$-measurable. Define $h : E \to F \times G$ by

$$h(x) = (f(x), g(x)) , \quad x \in E .$$

   Show that h is $\mathcal{E}/(\mathcal{F} \otimes \mathcal{G})$-measurable.

   [1 mark(s)]

2. Let $f : E \times F \to G$ be $(\mathcal{E} \otimes \mathcal{F})/\mathcal{G}$-measurable. For fixed $x_0 \in E$, show that the mapping $h : y \mapsto f(x_0, y)$ is $\mathcal{F}/\mathcal{G}$-measurable. The mapping h is called the *section* of f at x_0.

   Hint: Note that $h = f \circ g$, where $g : F \to E \times F, y \mapsto (x_0, y)$.

   [1 mark(s)]

3 Towards conditioning and sufficiency

Throughout this question, let $(\Omega, \mathcal{H}, P)$ be a probability space.

1. Recall that for $D \subset \Omega$,

$$\mathcal{D} = \mathcal{H} \cap D = \{A \cap D : A \in \mathcal{H}\}$$

   is a σ-algebra called the trace of $(\Omega, \mathcal{H})$ on D. Let $\mathcal{C}$ be a countable partition of Ω. Consider a set $C_n \in \mathcal{C}$ and the trace $\mathcal{C}_n$ of $(\Omega, \mathcal{H})$ on C_n. Define $\mu_n(A) = P(A)$ for any $A \in \mathcal{C}_n$, called the restriction of P to C_n. Show that μ_n is a measure, but not necessarily a probability measure, on $(C_n, \mathcal{C}_n)$. What is a necessary and sufficient condition for μ_n to be a probability measure on $(C_n, \mathcal{C}_n)$?

   [1 mark(s)]

2. Define a probability measure P_n on $(C_n, \mathcal{C}_n)$ constructed from μ_n. What must be true of $\mu_n(C_n)$ in order for this to be well-defined?

   [1 mark(s)]

3. P_n can be extended to be a probability measure $\hat{P}_n$ on $(\Omega, \mathcal{H})$ by defining $\hat{P}_n(A) = P_n(A \cap C_n)$ for any $A \in \mathcal{H}$.[1] Write down $\hat{P}_n(A)$, for any $A \in \mathcal{H}$, in terms of P. What would we call $\hat{P}_n(A)$ in an undergraduate course in probability?

   [1 mark(s)]

4. Show that our original probability measure, P, can be recovered as a convex combination, or *mixture*, of the set $\{\hat{P}_n\}_{n \geq 1}$.

   [3 mark(s)]

5. Let $(c_n)_{n \geq 1}$ be *mixing coefficients*, i.e., they satisfy $0 \leq c_n < 1$ and $\sum_{n \geq 1} c_n = 1$. Define $\nu(A) = \sum_{n \geq 1} c_n \hat{P}_n(A)$ for $A \in \mathcal{H}$. Show that ν is a probability measure on $(\Omega, \mathcal{H})$.

   [1 mark(s)]

[1] It's worth proving this to yourself for general measures; see Çinlar, Exercise I.3.12.

6. Now let ν be an arbitrary probability measure on $(\Omega, \mathcal{H})$. Define $\frac{0}{0} = 0$. Show that the restriction of ν to C_n is equal to that of P, for all $n \in \mathbb{N}$, that is,

$$\nu_n(A) = \frac{\nu(A)}{\nu(C_n)} = \frac{P(A)}{P(C_n)} = P_n(A) \ , \qquad \text{for all } A \in \mathcal{C}_n, \tag{1}$$

for all $n \in \mathbb{N}$, if and only if both of the following are true:

(a) $P(C_n) = 0 \Rightarrow \nu(C_n) = 0$.

(b) $\nu(A) = \sum_{n \geq 1} c_n \hat{P}_n(A)$ for each $A \in \mathcal{H}$, for some mixing coefficients $(c_n)_{n \geq 1}$.

[5 mark(s)]

Note: We will revisit these ideas in the context of conditioning.

Question total: [16 mark(s)]

Assignment total: [16 mark(s)]