STAT547C: Topics in Probability for Statistics **Winter Term 1 (Fall) 2021-22**

Assignment 3

Due date: Oct. 6, 11:59 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. You only have to do the required questions for full marks. While the other questions are optional, we encourage you to at least attempt them if you are motivated to learn the subject in depth.
2. *Justify your answers formally.*
3. Submit your work by 11:59 pm (Vancouver time) on the due date to Gradescope (.pdf) and Canvas (.tex).
4. Textbook questions marked **graded** and all non-textbook questions will be graded for correctness (partial credit will be given). The remaining questions will be marked on binary scale for "Did you make a good effort?"

1 Distribution functions

Required: Jacod and Protter (2nd edition!): 7.16 (**graded**), 7.18

[5 mark(s)]

2 Product spaces and sections

This exercise extends basic properties of measurable functions of one variable to functions of multiple variables. Let $(E, \mathcal{E}), (\mathcal{F}, \mathcal{F}), (\mathcal{G}, \mathcal{G})$ be measurable spaces.

A function that is measurable with respect to $\mathcal{E}$ and $\mathcal{F}$ is said to be $\mathcal{E}/\mathcal{F}$-measurable.

For sets $A \subset E$ and $B \subset F$, we let $A \times B = \{(x, y) \colon x \in A,\ y \in B\}$. If $A \in \mathcal{E}$ and $B \in \mathcal{F}$, then $A \times B$ is called a **measurable rectangle**. For the **product space** $E \times F$, the **product σ-algebra**, denoted $\mathcal{E} \otimes \mathcal{F}$, is the σ-algebra generated by the collection of measurable rectangles. The resulting measurable space is $(E \times F, \mathcal{E} \otimes \mathcal{F})$.

1. Let $f\colon E \to F$ be $\mathcal{E}/\mathcal{F}$-measurable, and $g\colon E \to G$ be $\mathcal{E}/\mathcal{G}$-measurable. Define $h\colon E \to F \times G$ by

$$h(x) = (f(x), g(x)) \ , \quad x \in E \ .$$

Show that h is $\mathcal{E}/(\mathcal{F} \otimes \mathcal{G})$-measurable.

[5 mark(s)]

2. Let $f\colon E \times F \to G$ be $(\mathcal{E} \otimes \mathcal{F})/\mathcal{G}$-measurable. For fixed $x_0 \in E$, show that the mapping $h\colon y \mapsto f(x_0, y)$ is $\mathcal{F}/\mathcal{G}$-measurable. The mapping h is called the *section* of f at x_0.

Hint: Note that $h = f \circ g$, where $g\colon F \to E \times F, y \mapsto (x_0, y)$.

[5 mark(s)]

3 Discrete σ-algebra and measurability

Suppose that a set E is countable, and take $\mathcal{E} = 2^E$, the discrete σ-algebra on E. Show that every function on E is $\mathcal{E}$-measurable.

[5 mark(s)]

4 Miscellaneous

Required: Jacod and Protter (2nd edition!): 9.2 (it might help to do 9.1 first)