STAT547C: Topics in Probability for Statistics **Winter Term 1 (Fall) 2021-22**

Assignment 2

Due date: Sept. 29, 11:59 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. You only have to do the required questions for full marks. While the other questions are optional, we encourage you to at least attempt them if you are motivated to learn the subject in depth.
2. *Justify your answers formally.*
3. Submit your work by 11:59 pm (Vancouver time) on the due date to Gradescope (.pdf) and Canvas (.tex).
4. Questions marked **Graded** will be graded for correctness (partial credit will be given). The remaining questions will be marked on binary scale for "Did you make a good effort?"

1 Random variables on a countable space

Required: Jacod and Protter (2nd edition!), 5.16, 5.20, 5.21

2 Restrictions and extensions

Graded

For this problem, we'll work with measures. A measure μ on a σ-algebra $\mathcal{E}$, $\mu\colon \mathcal{E} \to \mathbb{R}_+$, has the same properties as a probability measure, except that it is not required to have $\mu(\Omega) = 1$. We will assume throughout that it is finite: $\mu(\Omega) < +\infty$.

In particular: Let $(E, \mathcal{E})$ be a measurable space. A **measure** on $(E, \mathcal{E})$ is a mapping $\mu : \mathcal{E} \to \mathbb{R}_+$ the following properties:

a) *Zero on the empty set*: $\mu(\emptyset) = 0$.
b) *Countable additivity*: $\mu\big(\cup_n A_n\big) = \sum_n \mu(A_n)$ for every disjointed sequence (A_n) in $\mathcal{E}$.

The number $\mu(A) \in \mathbb{R}_+$ is called the measure of A. It is also written as μA.

1. Let $(E, \mathcal{E})$ be a measurable space. Fix $D \subset E$ and let

$$\mathcal{D} = \mathcal{E} \cap D = \{A \cap D : A \in \mathcal{E}\} .$$

Show that $\mathcal{D}$ is a σ-algebra on D. It is called the **trace** of $\mathcal{E}$ on D, and $(D, \mathcal{D})$ is called the trace of $(E, \mathcal{E})$ on D.

[4 mark(s)]

2. Let $(E, \mathcal{E})$ be a measurable space and μ a finite measure on it. Fix $D \in \mathcal{E}$.

   a) Define $\nu(A) = \mu(A \cap D)$, $A \in \mathcal{E}$. Show that ν is a measure on $(E, \mathcal{E})$. It is called the *trace* of μ on D.

   b) Let $\mathcal{D}$ be the trace of $\mathcal{E}$ on D. Define $\nu(A) = \mu(A)$ for $A \in \mathcal{D}$. Show that ν is a measure on $(D, \mathcal{D})$. It is called the *restriction* of μ to D.

[8 mark(s)]

3. Let $(E, \mathcal{E})$ be a measurable space, let $D \in \mathcal{E}$, and let $(D, \mathcal{D})$ be the trace of $(E, \mathcal{E})$ on D. Let μ be a measure on $(D, \mathcal{D})$ and define ν by

$$\nu(A) = \mu(A \cap D) , \quad A \in \mathcal{E} . \tag{1}$$

Show that ν is a measure on $(E, \mathcal{E})$. This device allows us to regard a "measure on D" as a "measure on E".

[4 mark(s)]

Assignment total: [16 mark(s)]