**STAT547C: Topics in Probability** **Winter Term 1 (Fall) 2020-21**

# Assignment 2

*Due date: Oct. 20 before class.* *Instructor: Benjamin Bloem-Reddy*

## Instructions

*Academic integrity policy:* I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. *Justify your answers formally.*
2. Submit your work before class on the due date to Gradescope (.pdf) and Canvas (.tex).

## 1 Towards conditioning and sufficiency

Throughout this question, let $(\Omega, \mathcal{H}, P)$ be a probability space.

(a) Suppose there is a random variable $X : \Omega \to \mathbb{R}$ taking values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that induces a measurable (countable) partition, $\mathcal{C} = (C_n)_{n\geq 1}$, of $\Omega$.

- What must be true of $X^{-1}A$ for each $A \in \mathcal{B}(\mathbb{R})$ in order for $X$ to be a random variable?
- In light of the previous problem, what must be true of $X$ on each member of $\mathcal{C}$?

**[2 mark(s)]**

(b) Recall that for $D \subset \Omega$,

$$\mathcal{D} = \mathcal{H} \cap D = \{A \cap D : A \in \mathcal{H}\}$$

is a $\sigma$-algebra called the trace of $(\Omega, \mathcal{H})$ on $D$. Let $\mathcal{C}$ be a countable partition of $\Omega$. Consider a set $C_n \in \mathcal{C}$ and the trace $\mathcal{C}_n$ of $(\Omega, \mathcal{H})$ on $C_n$. Define $\mu_n(A) = P(A)$ for any $A \in \mathcal{C}_n$, called the restriction of $P$ to $C_n$. Show that $\mu_n$ is a measure, but not necessarily a probability measure, on $(C_n, \mathcal{C}_n)$. What is a necessary and sufficient condition for $\mu_n$ to be a probability measure on $(C_n, \mathcal{C}_n)$?

**[4 mark(s)]**

(c) Define a probability measure $P_n$ on $(C_n, \mathcal{C}_n)$ constructed from $\mu_n$. What must be true of $\mu_n(C_n)$ in order for this to be well-defined?

**[2 mark(s)]**

(d) $P_n$ can be extended to be a probability measure $\hat{P}_n$ on $(\Omega, \mathcal{H})$ by defining $\hat{P}_n(A) = P_n(A \cap C_n)$ for any $A \in \mathcal{H}$.[1] Write down $\hat{P}_n(A)$, for any $A \in \mathcal{H}$, in terms of $P$. If $A$ and $C_n$ were random variables, what would we call $\hat{P}_n(A)$ in an undergraduate course in probability?

**[2 mark(s)]**

(e) Show that our original probability measure, $P$, can be recovered as a convex combination, or *mixture*, of the set $\{\hat{P}_n\}_{n\geq 1}$.

**[5 mark(s)]**

(f) Let $(c_n)_{n\geq 1}$ be *mixing coefficients*, i.e., they satisfy $0 \leq c_n < 1$ and $\sum_{n\geq 1} c_n = 1$. Define $\nu(A) = \sum_{n\geq 1} c_n \hat{P}_n(A)$ for $A \in \mathcal{H}$. Show that $\nu$ is a probability measure on $(\Omega, \mathcal{H})$.

**[3 mark(s)]**

(g) Now let $\nu$ be an arbitrary probability measure on $(\Omega, \mathcal{H})$. Define $\frac{0}{0} = 0$. Show that the restriction of $\nu$ to $C_n$ is equal to that of $P$, for all $n \in \mathbb{N}$, that is,

$$\nu_n(A) = \frac{\nu(A)}{\nu(C_n)} = \frac{P(A)}{P(C_n)} = P_n(A) \ , \qquad \text{for all } A \in \mathcal{C}_n, \tag{1}$$

for all $n \in \mathbb{N}$, if and only if both of the following are true:

(a) $P(C_n) = 0 \Rightarrow \nu(C_n) = 0$.

(b) $\nu(A) = \sum_{n\geq 1} c_n \hat{P}_n(A)$ for each $A \in \mathcal{H}$, for some mixing coefficients $(c_n)_{n\geq 1}$.

**[12 mark(s)]**

*Note*: We will revisit these ideas in the context of conditioning.

**Question total: [30 mark(s)]**

# 2 Everything you want to know about distribution functions

1. Let $F : \mathbb{R} \to [0, 1]$ be an increasing, right-continuous function, such that $\lim_{t\to\infty} F(t) = F(+\infty) = 1$ and $\lim_{t\to-\infty} F(t) = F(-\infty) = 0$. Show that $F$ has at most a countable number of discontinuities (jumps).

**[5 mark(s)]**

2. Define

$$Q(u) = \inf\{t \in \mathbb{R} : F(t) > u\} \ , \quad u \in (0, 1) \ , \tag{2}$$

with the usual convention that $\inf \emptyset = \infty$. Show that the function $Q : (0, 1) \to \bar{\mathbb{R}}$ is increasing and right-continuous.

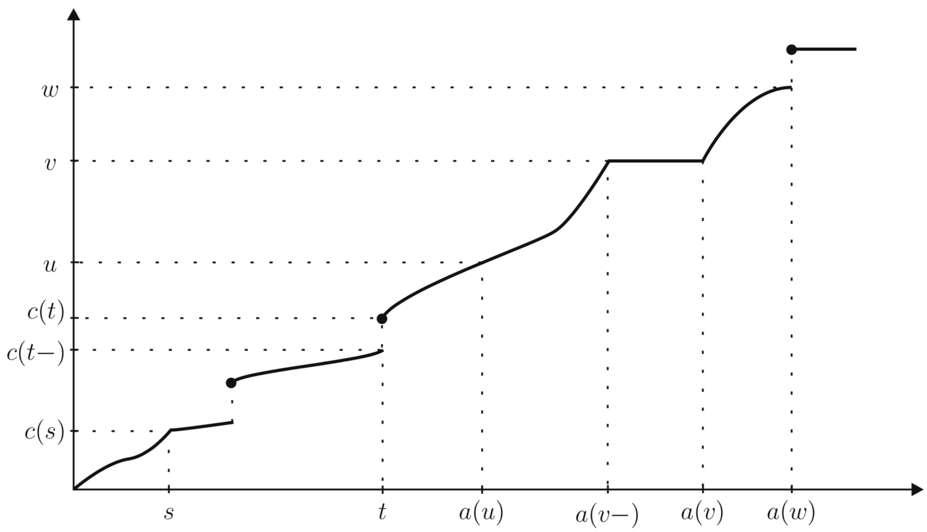
Convince yourself (see the figure below (with $c = F$ and $a = Q$), taken from Çinlar, *Probability and Stochastics*) that $F$ and $Q$ are right-continuous "functional inverses" of each other. That is,

$$F(t) = \inf\{u \in (0, 1) : Q(u) > t\} \ . \tag{3}$$

(You do not need to prove this.)

**[5 mark(s)]**

[1] It's worth proving this to yourself for general measures; see Çinlar, Exercise I.3.12.




3. Show that $Q(F(t)) \geq t$, with equality if and only if $F(t+\epsilon) > F(t)$ for every $\epsilon > 0$.

**[5 mark(s)]**

4. Let $\mu$ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(t) = \mu[-\infty, t]$ is finite for every $t \in \mathbb{R}$ and $F(\infty) = \lim_{t\to\infty} F(t) = 1$.

   i) Show that $F$ is increasing and right-continuous.

   **[2 mark(s)]**

   ii) Define $Q(u)$ as above for $u \in (0,1)$, and let $\lambda$ denote the Lebesgue measure on $(0,1)$. Show that

   $$\mu = \lambda \circ Q^{-1}$$

   **[3 mark(s)]**

   iii) Let $X$ be a $\mathbb{R}$-valued random variable with distribution $\mu$. Then $F$ is called its *distribution function* and $Q$ its *quantile function* because

   $$\mathbb{P}\{X \leq Q(u)\} = u \ , \quad u \in (0,1). \tag{4}$$

   For simplicity, assume that $F$ is continuous, i.e., it has no jumps. Let $U$ be a random variable with uniform distribution on $(0,1)$, and $Y = Q \circ U$. Show that $Y \overset{d}{=} X$.

   **[5 mark(s)]**

5. Let $X$ be a random variable taking values in $\bar{\mathbb{R}} = [-\infty, \infty]$. Let $\mu$ be its distribution, and $F : \mathbb{R} \to [0,1]$ its distribution function as defined above. Define the left-hand limit $F(x-) = \lim_{t\uparrow x} F(t)$, which exists for every $x \in \mathbb{R}$ because $F$ is increasing. Similarly, the limits $F(-\infty) = \lim_{x\downarrow -\infty} F(x)$ (not necessarily equal to zero) and $F(+\infty) = \lim_{x\uparrow\infty}(x)$ (not necessarily equal to one) exist.

   Let $D$ be the set of all atoms of the distribution $\mu$. Then $D$ consists of all $x \in \mathbb{R}$ for which $F(x) - F(x_-) > 0$, plus the point $-\infty$ if $F(-\infty) > 0$, plus the point $+\infty$ if $F(+\infty) < 1$. As established above, $D$ is countable. Define $D_x = D \cap (-\infty, x]$ and

   $$F_d(x) = F(-\infty) + \sum_{t \in D_x} (F(t) - F(t_-)) \ , \quad F_c(x) = F(x) - F_d(x) \ , \quad x \in \mathbb{R} \ . \tag{5}$$

Then $F_d$ is an increasing right-continuous function that increases by jumps only, and $F_c$ is increasing continuous. Show that $F_d$ is the distribution function of the measure

$$\mu_d(B) = \mu(B \cap D) , \quad B \in \mathcal{B}(\bar{\mathbb{R}}) , \tag{6}$$

and $F_c$ is the distribution function of the measure $\mu_c = \mu - \mu_d$.

**[5 mark(s)]**

**Question total: [30 mark(s)]**

# 3 Product spaces and sections

This exercise extends basic properties of measurable functions of one variable to functions of multiple variables. Let $(E, \mathcal{E}), (\mathcal{F}, \mathcal{F}), (\mathcal{G}, \mathcal{G})$ be measurable spaces.

1. Let $f\colon E \to F$ be $\mathcal{E}/\mathcal{F}$-measurable, and $g\colon E \to G$ be $\mathcal{E}/\mathcal{G}$-measurable. Define $h\colon E \to F \times G$ by

   $$h(x) = (f(x), g(x)) , \quad x \in E .$$

   Show that $h$ is $\mathcal{E}/(\mathcal{F} \otimes \mathcal{G})$-measurable.

   **[10 mark(s)]**

2. Let $f\colon E \times F \to G$ be $(\mathcal{E} \otimes \mathcal{F})/\mathcal{G}$-measurable. For fixed $x_0 \in E$, show that the mapping $h\colon y \mapsto f(x_0, y)$ is $\mathcal{F}/\mathcal{G}$-measurable. The mapping $h$ is called the *section* of $f$ at $x_0$.

   *Hint*: Note that $h = f \circ g$, where $g\colon F \to E \times F, y \mapsto (x_0, y)$.

   **[10 mark(s)]**

**Question total: [20 mark(s)]**

**Assignment total: [80 mark(s)]**