STAT547C: Topics in Probability **Winter Term 1 (Fall) 2020-21**

Assignment 1

Due date: Sept. 29 before class. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. *Justify your answers formally.*
2. Submit your work before class on the due date to Gradescope (.pdf) and Canvas (.tex).

1 Basics of measure theory

In all of the following, let E be a set.

1. Let $\mathcal{A} = A \subsetneq E$, a single set. What is $\sigma\mathcal{A}$?

   [3 mark(s)]

2. Give an example of two σ-algebras, $\mathcal{E}_1$ and $\mathcal{E}_2$, for which $\mathcal{E}_1 \cup \mathcal{E}_2$ is not a σ-algebra.

   [5 mark(s)]

3. Prove Proposition 1.2: Let $\mathcal{C}$ and $\mathcal{D}$ be two collections of subsets of E. Then:
   (a) If $\mathcal{C} \subset \mathcal{D}$ then $\sigma\mathcal{C} \subset \sigma\mathcal{D}$.
   (b) If $\mathcal{C} \subset \sigma\mathcal{D}$ then $\sigma\mathcal{C} \subset \sigma\mathcal{D}$.
   (c) If $\mathcal{C} \subset \sigma\mathcal{D}$ and $\mathcal{D} \subset \sigma\mathcal{C}$, then $\sigma\mathcal{C} = \sigma\mathcal{D}$.
   (d) If $\mathcal{C} \subset \mathcal{D} \subset \sigma\mathcal{C}$, then $\sigma\mathcal{C} = \sigma\mathcal{D}$.

   [5 mark(s)]

4. Let $\mathcal{E}$ be a σ-algebra on E. If $f_1 \colon E \to \mathbb{R}$ and $f_2 \colon E \to \mathbb{R}$ are $\mathcal{E}$-measurable functions, show that $f' \colon E \to \mathbb{R}, x \mapsto \max\{f_1(x), f_2(x)\}$ is also $\mathcal{E}$-measurable.

   [5 mark(s)]

5. Let $\mathcal{E}$ be a σ-algebra on E. If $f_1 \colon E \to \mathbb{R}$ and $f_2 \colon E \to \mathbb{R}$ are $\mathcal{E}$-measurable functions, show that $f' \colon E \to \mathbb{R}, x \mapsto \min\{f_1(x), f_2(x)\}$ is also $\mathcal{E}$-measurable.

   [5 mark(s)]

Question total: [23 mark(s)]

2 Partition-generated σ-algebras

(a) Let $\mathcal{D} = \{A, B, C\}$ be a partition of E. List the elements of $\sigma\mathcal{D}$.

[5 mark(s)]

(b) Let $\mathcal{C}$ be a countable partition of E. Show that every element of $\sigma\mathcal{C}$ is a countable union of elements taken from $\mathcal{C}$.

Hint: Let $\mathcal{S}$ be the collection of of all sets that are countable unions of elements taken from $\mathcal{C}$. Show that $\mathcal{S}$ is a σ-algebra, and argue that $\mathcal{S} = \sigma\mathcal{C}$.

Hint: You might find it useful to recall the fact that the union of a countable collection of sets, each of which is countable, is again countable.

[15 mark(s)]

(c) A σ-algebra is **countably generated** if it is generated by countably many sets. Argue that the Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is countably generated.

Hint: There is not much to prove; we essentially answered this in an exercise in class.

[2 mark(s)]

(d) Suppose that $\mathcal{E}$ is generated by a countable partition $\mathcal{C}$ of E. Show that in this case, a $\mathbb{R}$-valued function is $\mathcal{E}$-measurable if and only if it is constant on each member of that partition.

[10 mark(s)]

Question total: [32 mark(s)]

3 Continuous functions

Suppose that E is a toplogocial space. Show that every continuous function $f : E \to \mathbb{R}$ is a Borel function (i.e., it is $\mathcal{B}(E)/\mathcal{B}(\mathbb{R})$-measurable).

Hint: If f is continuous, then $f^{-1}B$ is open for every open subset $B \subset \mathbb{R}$.

[10 mark(s)]

Question total: [10 mark(s)]

4 Functional representation of measurable functions

Recall that for a set E and a measurable space $(F, \mathcal{F})$, the σ-algebra $f^{-1}\mathcal{F}$ on E generated by a function $f : E \to F$ is $f^{-1}\mathcal{F} = \{f^{-1}B : B \in \mathcal{F}\}$.

Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be measurable spaces, and $(F, \mathcal{B}(F))$ a standard Borel space. Fix two $\mathcal{E}$-measurable functions $f : E \to F$ and $g : E \to G$. Show that f is $g^{-1}\mathcal{G}$-measurable (i.e., $f^{-1}\mathcal{F} \subset g^{-1}\mathcal{G}$) if and only if there exists some $\mathcal{G}$-measurable mapping $h : G \to F$ such that $f = h \circ g$.

Hint: First, prove for indicator and simple functions, then for positive and arbitrary functions.

[15 mark(s)]

Question total: [15 mark(s)]

Assignment total: [80 mark(s)]