STAT547C: Topics in Probability **Fall 2019-20**

Assignment 3: Supplements to conditioning

Due date: Nov. 20 by 3 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeX equation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. You only have to do the non-optional questions for full marks. While the other questions are optional, we encourage you to at least attempt them if you are motivated to learn the subject in depth. Effort to do so is one of the ways to get participation points (the main other ways are the exercises, the learning logs, and participation during the lecture and the office hours).
2. *Justify your answers formally.*

Throughout this assignment, let $(\Omega, \mathcal{H}, \mathbb{P})$ be a (background) probability space.

1 Conditional densities

A topic we did not cover in lecture is **conditional density**. Suppose that the joint distribution π of X and Y (random variables taking values in $(D, \mathcal{D})$ and $(E, \mathcal{E})$, respectively) has the form

$$\pi(dx, dy) = \mu_0(dx)\nu_0(dy)p(x, y) \ , \quad x \in D, \ y \in E \ ,$$

where μ_0 and ν_0 are σ-finite measures and p is a positive function that belongs to $\mathcal{D} \otimes \mathcal{E}$. (Often, $D = E = \mathbb{R}^D$ and $\mu_0 = \nu_0$ =Lebesgue.) This π can be put in the form (10.20) from the lecture notes:

$$\pi(dx, dy) = \mu(dx)K(x, dy) = [\mu_0(dx)m(x)][\nu_0(dy)k(x, y)] \ , \tag{1}$$

with

$$m(x) = \int_E \nu_0(dy)p(x, y) \ , \quad k(x, y) = \begin{cases} p(x, y)/m(x) \ , & m(x) > 0 \ , \\ \int_D \mu_0(dx')p(x', y) \ , & m(x) = 0 \ . \end{cases} \tag{2}$$

Then the function $y \mapsto k(x, y)$ is called the **conditional density** (with respect to ν_0) of Y given that $X = x$.

To illustrate, consider independent gamma random variables $Y \overset{d}{=} \gamma_{a,c}$ and $Z \overset{d}{=} \gamma_{b,c}$. Let $X = Y + Z$.

(a) What is the joint distribution of X and Y, $\pi(dx, dy)$?

[10 mark(s)]

(b) What is the distribution of X? What is its density with respect to the Lebesgue measure?

[10 mark(s)]

(c) Let μ denote the distribution of X. The conditional density of Y given $X = x$ is obtained from the kernel

$$K(x, dy) = \frac{\pi(dx, dy)}{\mu(dx)} \ . \tag{3}$$

What is K? What is k? In particular, what is K when $a = b = 1$ (corresponding to exponential random variables)?

[10 mark(s)]

Question total: [30 mark(s)]

2 Sufficiency & co.

Assume that $(\Omega, \mathcal{H})$ is standard. Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be a family of probability measures on $(\Omega, \mathcal{H})$ indexed by Θ (i.e., it is a statistical model). The family $\mathcal{P}$ is **dominated** by a measure ν if $P_\theta \ll \nu$ for each $P_\theta \in \mathcal{P}$.

a) **Sufficient σ-algebras.** We talked in class about sufficient statistics. The measure-theoretic definition of sufficiency deals with sufficient σ-algebras. Specifically, a sub-σ-algebra $\mathcal{F}$ is sufficient for $\mathcal{P}$ if there is a transition probability kernel K from $(\Omega, \mathcal{H})$ into $(\Omega, \mathcal{H})$ such that $K(\cdot, A)$ is a (regular) version of $P_\theta[A \mid \mathcal{F}]$ for all $A \in \mathcal{H}$, $\theta \in \Theta$ (that is, $K(\omega, A) = P_\theta[A \mid \mathcal{F}](\omega)$ for each $\omega \in \Omega$).

Suppose that $\mathcal{P}$ is dominated by a probability measure P_* and that each P_θ has a density f_θ with respect to P_*. Furthermore, assume that f_θ belongs to $\mathcal{F}$. Show that $P_*[A \mid \mathcal{F}]$ is a version of $P_\theta[A \mid \mathcal{F}]$ for each $A \in \mathcal{E}$ and $\theta \in \Theta$, and therefore $\mathcal{F}$ is sufficient for $\mathcal{P}$.

[15 mark(s)]

b) Let $S : E \to F$ be $\mathcal{E}/\mathcal{F}$-measurable, and define

$$K(x, B) = \mathbf{1}_B \circ S(x) \ , \quad x \in E, \ B \in \mathcal{F} \ . \tag{4}$$

Show (by direct computation) that, using the notation of Theorem 10.7 in the lecture notes,

i) $Kf = f \circ S$ for $f : F \to \mathbb{R}_+$ belonging to $\mathcal{F}_+$.

ii) $\mu K = \mu \circ S^{-1}$ is a measure on $(F, \mathcal{F})$ for every measure μ on $(E, \mathcal{E})$.

iii) $\mu K f = \mu(f \circ S)$ for every measure μ on $(E, \mathcal{E})$ and f belonging to $\mathcal{F}_+$.

[15 mark(s)]

c) As defined above, $S : E \to F$ is a statistic of X, and part a) holds for any statistic. We're interested in sufficient statistics. For simplicity, let $(F, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let σS be sufficient for the model $\mathcal{P}$, which we further assume to be dominated by P_*. By part a), each P_θ has density with respect to P_*

$$p_\theta(x) = f_\theta(x) h(x) \ , \quad \text{with } f_\theta \text{ belonging to } \sigma S. \tag{5}$$

For $s \in \mathbb{R}$, let $S^{-1}\{s\} = \{x \in E : S(x) = s\}$.

Show that the conditional distribution of X given S is given by (for all $\theta \in \Theta$, with the convention that $\frac{0}{0} = 0$),

$$\hat{K}(s, dx) = \frac{P_*(dx)h(x)}{\int_{S^{-1}\{s\}} P_*(dx')h(x')} \mathbf{1}_{S^{-1}\{s\}}(x) \, . \tag{6}$$

[15 mark(s)]

Question total: [45 mark(s)]

Assignment total: [75 mark(s)]