STAT547C: Topics in Probability **Fall 2019-20**

Assignment 2

Due date: Oct. 11 by 5 pm. *Instructor: Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a LaTeXequation is not permitted; and if you use online resources, you must cite them. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. You only have to do the non-optional questions for full marks. While the other questions are optional, we encourage you to at least attempt them if you are motivated to learn the subject in depth. Effort to do so is one of the ways to get participation points (the main other ways are the exercises, the learning logs, and participation during the lecture and the office hours).
2. *Justify your answers formally.*

Throughout this assignment, let $(\Omega, \mathcal{H}, \mathbb{P})$ be a (background) probability space.

1 Everything you want to know about distribution functions

(a) Let $F : \mathbb{R} \to [0, 1]$ be an increasing, right-continuous function, such that $\lim_{t\to\infty} F(t) = F(+\infty) = 1$ and $\lim_{t\to-\infty} F(t) = F(-\infty) = 0$. Show that F has at most a countable number of discontinuities (jumps).

[5 mark(s)]

(b) Define

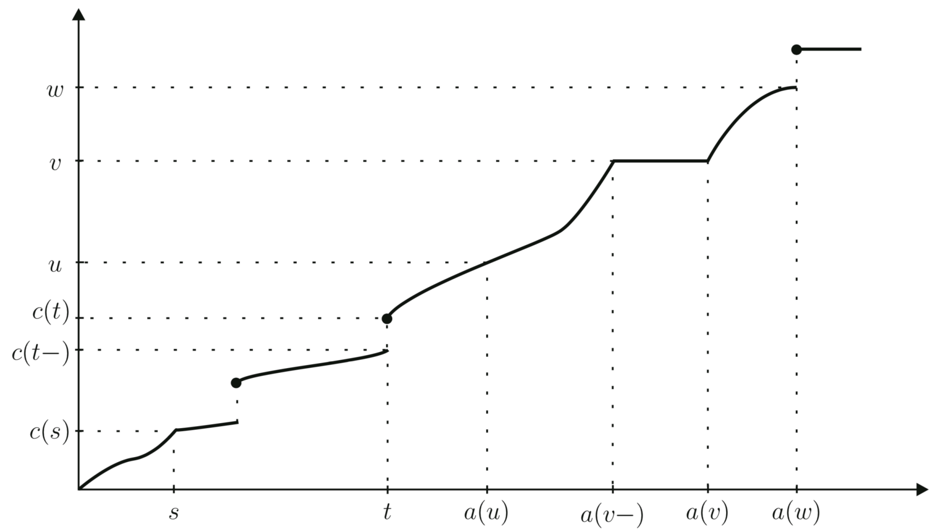
$$Q(u) = \inf\{t \in \mathbb{R} : F(t) > u\} \ , \quad u \in (0, 1) \ , \tag{1}$$

with the usual convention that $\inf \emptyset = \infty$. Show that the function $Q : (0, 1) \to \bar{\mathbb{R}}$ is increasing and right-continuous, and that

$$F(t) = \inf\{u \in (0, 1) : Q(u) > t\} \ . \tag{2}$$

Thus, F and Q are right-continuous "functional inverses" of each other. (See the figure below (with $c = F$ and $a = Q$), taken from Çinlar, *Probability and Stochastics*.)

[5 mark(s)]



(c) Show that $Q(F(t)) \geq t$, with equality if and only if $F(t+\epsilon) > F(t)$ for every $\epsilon > 0$.

[5 mark(s)]

(d) Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $F(t) = \mu[-\infty, t]$ is finite for every $t \in \mathbb{R}$ and $F(\infty) = \lim_{t\to\infty} F(t) = 1$.

i) Show that F is increasing and right-continuous.

[2 mark(s)]

ii) Define $Q(u)$ as above for $u \in (0,1)$, and let λ denote the Lebesgue measure on $(0,1)$. Show that

$$\mu = \lambda \circ Q^{-1}$$

[3 mark(s)]

iii) Let X be a $\mathbb{R}$-valued random variable with distribution μ. Then F is called its *distribution function* and Q its *quantile function* because

$$\mathbb{P}\{X \leq Q(u)\} = u\ , \quad u \in (0,1). \tag{3}$$

For simplicity, assume that F is continuous, i.e., it has no jumps. Let U be a random variable with uniform distribution on $(0,1)$, and $Y = Q \circ U$. Show that $Y \overset{d}{=} X$.

[5 mark(s)]

(e) Let X be a random variable taking values in $\bar{\mathbb{R}} = [-\infty, \infty]$. Let μ be its distribution, and $F : \mathbb{R} \to [0,1]$ its distribution function as defined above. Define the left-hand limit $F(x-) = \lim_{t\uparrow x} F(t)$, which exists for every $x \in \mathbb{R}$ because F is increasing. Similarly, the limits $F(-\infty) = \lim_{x\downarrow -\infty} F(x)$ (not necessarily equal to zero) and $F(+\infty) = \lim_{x\uparrow\infty}(x)$ (not necessarily equal to one) exist.

Let D be the set of all atoms of the distribution μ. Then D consists of all $x \in \mathbb{R}$ for which $F(x) - F(x-) > 0$, plus the point $-\infty$ if $F(-\infty) > 0$, plus the point $+\infty$ if $F(+\infty) < 1$. As established above, D is countable. Define $D_x = D \cap (-\infty, x]$ and

$$F_d(x) = F(-\infty) + \sum_{t\in D_x} (F(t) - F(t-))\ , \quad F_c(x) = F(x) - F_d(x)\ , \quad x \in \mathbb{R}\ . \tag{4}$$

Then F_d is an increasing right-continuous function that increases by jumps only, and F_c is increasing continuous. Show that F_d is the distribution function of the measure

$$\mu_d(B) = \mu(B \cap D) , \quad B \in \mathcal{B}(\bar{\mathbb{R}}) , \tag{5}$$

and F_c is the distribution function of the measure $\mu_c = \mu - \mu_d$.

[5 mark(s)]

Question total: [30 mark(s)]

2 Simplifiable random variables and beta-gamma algebra

(a) Let X, Y, Z be random variables taking values in $\mathbb{R}_+$, with $X \perp\!\!\!\perp Y$ and $Z \perp\!\!\!\perp Y$. Y is said to be **simplifiable** if $XY \stackrel{d}{=} YZ$ implies $X \stackrel{d}{=} Z$.

Prove that, if $\mathbb{P}(Y = 0) = 0$ and if the characteristic function of $\ln Y$ has only isolated zeros (i.e., every zero has a neighborhood that does not contain any other zeros), then Y is simplifiable.

Warning! $\ln X$ is not well-defined when $X = 0$, which the hypothesis does not rule out (likewise for $\ln Z$).

[5 mark(s)]

(b) Let $\gamma_{a,b}$ be a gamma random variable with scale parameter a and rate parameter b, and $\beta_{a,b}$ a beta random variable with parameters a and b. That is,

$$\mathbb{P}(\gamma_{a,b} \in dx) = \lambda(dx) x^{a-1} e^{-bt} b^{-a} / \Gamma(a) , \quad x \in \mathbb{R}_+ ,$$

and

$$\mathbb{P}(\beta_{a,b} \in dx) = \lambda(dx) \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} , \quad x \in [0,1] .$$

Let γ_a be a gamma random variable with scale parameter a, and $b = 1$.

We saw in class that for $\gamma_a \perp\!\!\!\perp \gamma_b$,

$$\left(\gamma_a + \gamma_b, \frac{\gamma_a}{\gamma_a + \gamma_b} \right) \stackrel{d}{=} (\gamma_{a+b}, \beta_{a,b}) ,$$

with independence between the elements on the left-hand side, and also independence between the elements on the right-hand side.

Prove a different version of this: for $\gamma_a \perp\!\!\!\perp \gamma_b$,

$$(\gamma_a, \gamma_b) \stackrel{d}{=} (\beta_{a,b}\gamma_{a+b}, (1 - \beta_{a,b})\gamma_{a+b}) , \tag{6}$$

with $\beta_{a,b} \perp\!\!\!\perp \gamma_{a+b}$.

[5 mark(s)]

(c) Show the identity (with independence on the right-hand side)

$$\beta_{a,b+c} \stackrel{d}{=} \beta_{a,b}\beta_{a+b,c} . \tag{7}$$

Hint: Use the fact that γ_{a+b+c} is simplifiable.

[7 mark(s)]

(d) Fix some $m \in \mathbb{N}_+$. Let $(\gamma_1^j)_{j\geq 1}$ and $(\gamma_{m-1}^j)_{j\geq 2}$ be sequences of i.i.d. gamma random variables, also independent of each other. For $j > 1$, let

$$X_j = \frac{\gamma_1^j}{\sum_{i=2}^j \gamma_{m-1}^i + \sum_{i=1}^j \gamma_1^i} \ . \tag{8}$$

i) Using the identities above, show that $(X_j)_{j\geq 2}$ is a sequence of mutually independent beta random variables, equal in distribution to $(\beta_{1,m(j-1)})_{j\geq 2}$.

[5 mark(s)]

ii) Define $\eta_j := m(j-1)+1$. Show that for any $\epsilon > 0$,

$$\mathbb{P}\big[|X_j - \gamma_1^j/\eta_j| \geq \epsilon\big] \leq \frac{1}{\epsilon^2}\frac{2}{\eta_j^2(\eta_j+1)} \ . \tag{9}$$

[8 mark(s)]

Question total: [30 mark(s)]

3 Size bias (optional)

(a) Let X be a random variable taking values in $\mathbb{R}_+$, with $\mathbb{E}[X] \in (0,\infty)$. Let its distribution be μ. Let X^* be another random variable with distribution $\nu \ll \mu$, such that

$$\frac{d\nu}{d\mu}(x) = \frac{\mathbb{P}(X^* \in dx)}{\mathbb{P}(X \in dx)} = \frac{x}{\mathbb{E}[X]} \ . \tag{10}$$

X^* is said to be a *size-biased* version of X, and to have the size-biased X distribution. Assume that μ is absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{R}_+$, so that X has density f_X. What is the density of X^*, f_{X^*}?

[5 mark(s)]

(b) Let X be a random variable with Poisson(α) distribution. Show that $X^* \overset{d}{=} X + 1$.

[5 mark(s)]

(c) Let $\gamma_{a,b}$ be a gamma random variable, as above. What is the distribution of the size-biased version $\gamma_{a,b}^*$?

[5 mark(s)]

(d) Let $\beta_{a,b}$ be a beta random variable, as above. What is the distribution of the size-biased version $\beta_{a,b}^*$?

[5 mark(s)]

(e) Let $\beta_{a,1}$ be a beta random variable, and $\beta_{a,1}^*$ a size-biased version such that $\beta_{a,1} \perp\!\!\!\perp \beta_{a,1}^*$. What is the distribution of $\beta_{a,1}\beta_{a,1}^*$?

[5 mark(s)]

Assignment total: [60 mark(s)]

Extra credit total: [25 mark(s)]